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Understanding-Oriented Mathematics Instruction using the Example of Solving a Word Problem

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Abstract In the framework of the Swiss-German study "Quality of instruction, learning behaviour and mathematical understanding" (Klieme et al. Dokumentation der Erhebungs- und Auswertungsinstrumente zur schweizerisch-deutschen Videostudie ‚Unterrichtsqualität, Lernverhalten und mathematisches Verständnis', Vols. 13–15, Chaps. 1–3, 2005, 2006a, 2006b), using the example of a word problem, it was examined in 37 classes how teachers support the problem-solving process in classroom instruction from a subject-based and communicative perspective. To this aim, an analysis instrument was developed which describes both content-related aspects of the problem-solving process and the students' participation in this regard. A central part of the instrument measures the extent to which understanding on the level of a situation model and/or a mathematical model is explicitly addressed. The results highlight that the individual teachers differ in terms of the type of their support behaviour during the modelling process, and they can be categorised into different profiles of understanding-oriented and didactically supportive handling of a word problem in classroom instruction.

Keywords Word problems · Modelling process · Video analysis · Didactic communication · Supporting understanding

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Verstehensorientiertes Arbeiten im Mathematikunterricht am Beispiel des Lösens einer Textaufgabe

Zusammenfassung Im Rahmen der schweizerisch-deutschen Studie „Unterrichtsqualität, Lernverhalten und mathematisches Verständnis" (Klieme et al. Dokumentation der Erhebungs- und Auswertungsinstrumente zur schweizerisch-deutschen Videostudie ‚Unterrichtsqualität, Lernverhalten und mathematisches Verständnis', Vols. 13–15, Chaps. 1–3, 2005, 2006a, 2006b) wurde in 37 Klassen am Beispiel einer Textaufgabe untersucht, wie Lehrpersonen den Problemlösungsprozess im Klassenunterricht fachlich und kommunikativ unterstützen. Dazu wurde ein Analyseinstrument entwickelt, das sowohl inhaltliche Aspekte des Lösungsprozesses als auch die diesbezügliche Partizipation der Schüler beschreibt. Ein zentraler Teil des Instrumentes erfasst, inwiefern Verständnis auf der Ebene eines Situationsmodells bzw. eines mathematischen Modells explizit thematisiert wird. Die Ergebnisse machen deutlich, dass sich die einzelnen Lehrpersonen in der Art ihres Unterstützungsverhaltens während des Modellierungsprozesses unterscheiden. Sie lassen sich verschiedenen Profilen des verstehensorientierten und didaktisch unterstützenden Umgangs mit einer Textaufgabe im Klassenunterricht zuordnen.

1 Introduction

There is broad consensus that the fostering of understanding is an important goal of mathematics instruction (e.g. Freudenthal 1977; Hiebert et al. 1997). But what does this actually mean in concrete terms? How can mathematics instruction be fostered in an optimal manner?

1.1 Change in the Task Culture and Analysis of Learning Processes

Particularly since the results of TIMSS, PISA etc., there has been a strong research interest in this regard, both through in-depth analysis of everyday instructional practice and through scientifically grounded reform initiatives, for instance in the framework of SINUS and DISUM, which aim to bring about concrete improvements in this practice (e.g. Baptist and Raab 2007; Leiss et al. 2008). The change in task culture represents a central focus within these endeavours. The aim is to bring about problem-setting that is increasingly oriented towards reality (modelling tasks) during instruction, as this requires and demands application-oriented problem-solving processes, or *mathematical modelling* (z. B. Blum et al. 2006; Borromeo Ferri and Kaiser 2008; Büchter and Leuders 2005; Greer et al. 2007; Maass 2004, 2007). In addition to the *change in the culture of tasks* as an optimisation of the building up of mathematical competence, the *analysis of understanding-oriented teaching and learning processes* has garnered increasing attention.

The video-based analysis of mathematics instruction in lower secondary-level classes from Germany and Switzerland (also known as the "Pythagoras project"), upon which the current article is based, tied in with the earlier video studies within TIMSS, and pursues the goal of enabling more in-depth analysis of teaching processes. In this respect, besides general didactical and psychological research questions, subject-didactical perspectives are also taken into account. To be able to carry out comparative microanalyses of teaching and learning processes, in the project, a total of 5 lessons per teacher were videotaped on a standardised learning content. Two instructional units were recorded as the object of investigation, which represent two typical learning situations: On the one hand an instructional unit on the introduction to the Pythagorean Theorem, and on the other hand a unit on solving mathematical word problems on the basis of a set of tasks provided by the study leaders.[1] Central to the analyses in terms of the latter was the support given to the students when solving problems. The current article concentrates on this instructional unit on solving word problems.

1.2 Word Problems as an Object of Investigation

An instructional unit on solving word problems was included in the video study for several reasons. First of all, the concern was with a type of task which—although much criticized as an object of instruction—continues to arise relatively frequently in mathematics instruction and can thus be seen as an example of a teaching and learning situation.[2] As many students find traditional word problems difficult to solve, supporting the solving of such problems in an appropriate way places particular demands on teachers. The question of effective forms of learning support in relation to solving word problems is also seen against the background of results of comprehensive research on the solving of mathematical word problems, which has repeatedly pointed out characteristic difficulties and student errors when solving word problems (cf. in particular Cummins-Dellarosa et al., 1988; Reusser 1985; Reusser and Stebler 1997; Van Dooren et al. 2006; Verschaffel et al. 1999). Besides diverse linguistic, situation-based and mathematics-based difficulty factors which prove to be important to successfully solve a task, it has been shown that students very frequently still “solve” tasks even when they do not allow any meaningful solution (cf. among others Baruk 1989; Reusser 1988; Reusser 1997; Reusser and Stebler 1997). The reason for such faulty solutions is assumed to lie on the one hand in context factors and on the other in unfavourable problem-solving strategies, in particular a procedure that has been described on various occasions as “superficial modelling” (among others Verschaffel et al. 2000, p. 13; Van Dooren et al. 2006, p. 96). It essentially consists of a central step of understanding-based problem-solving—the situation understanding—being skipped, passing over to a

[1] The tasks were traditional word problems with limited potential for everyday-world modelling.

[2] In the sample of the Pythagoras study ($N = 40$), the evaluation of the teacher questionnaire also shows that word problems continue to be widespread in instruction. In response to the question of what proportion of the teaching time is taken up by word problems in their lessons, only 17.5% of teachers indicated using less than 20% of the teaching time for solving word problems. For 45.5% of teachers, this figure lay at between 20 and 40%, for 30% of teachers between 40 and 60% and for 5% of teachers more than 60% of teaching time.

more or less plausible operation with numbers that occur in the word text ("direct translation" in the sense of Paige and Simon 1966, cf. Reusser 1997).

The more precise reasons for this student behaviour are to be sought on the one hand in the quality of the tasks, but on the other hand also in how the mathematical word problems are dealt with in instructional terms in the classroom (quality of working on tasks). With respect to the *quality of tasks*, the criticism can be levelled that many such tasks, through their often artificial character and failure to correspond to reality, virtually call out for the application of superficial problem-solving strategies like those described above (cf. among others Reusser and Stebler 1997; Schoenfeld 1982). In terms of *dealing with word problems* in everyday mathematics instruction, by contrast, comparatively few research results are available thus far. The importance of a conducive learning culture has received particular emphasis, including appropriate beliefs on the part of the learners (cf. Van Dooren et al. 2006; Verschaffel et al. 2002).

So far, it has barely been investigated how teachers deal with word problems in mathematics instruction, for instance how they support the students' processes of understanding and problem-solving in a concrete manner. In view of the aforementioned research findings, a suitable support in solving mathematical word problems during lessons should in particular also counteract the application of unfavourable problem-solving strategies such as "superficial modelling" (Verschaffel et al. 2000). Consequently, an *understanding-oriented support* appears to be important here, which avoids an immediate mathematization without a thorough understanding, and instead initially strives to achieve an understanding of the conditions and relationships as a first and necessary step, and only on this basis undertakes the implementation in a mathematical model.

1.3 Aims of the Analysis

The aim of the analysis presented below was to examine, from a comparative perspective, the quality of working on word problem typical for the school level examined, on a whole-class basis. To this aim, an instrument was developed which, on the basis of a process model of solving mathematical word problems, allows for the specific demands of such tasks (in particular the distinction between situation understanding and mathematical model). Moreover, the quality of the classroom discourse was taken into account with regard to the extent to which students were able to make substantial contributions to participating in the problem-solving process.

With the help of the instrument, in the framework of the CH-D video study (Klieme et al. 2005, 2006a, 2006b), from a total of 37 classes from the 8th/9th school year from Germany and Switzerland, one videotaped instructional sequence from each class was analysed, in which a standardised mathematical word problem was solved as a class.

In the following, the theoretical background is first briefly presented, and the methodological procedure is elucidated. In section four, several selected results of the study are presented. In the final section, the findings are discussed in summary and consequences for the further education and training of teachers are formulated.

2 Theoretical Background

2.1 Word Problems as Learning Opportunities for Elementary Linguistic-Mathematical Problem-Solving

In the literature, several theoretical models can be found, which describe the complex process of understanding word problems as a mental linguistic-mathematical process of construction and modelling. For the current study, the multi-level model SPS (SituationProblemSolver) of Reusser (1985, 1989), and the model of Blum et al. (2004) are particularly important.

The cyclical model of understanding and solving mathematical word problems SPS[3] of Reusser, with its cognitive psychological foundation, can be understood as a "psychological framework theory of mathematization and as a psychological-didactic process model of understanding and solving mathematical word and situation problems" (Reusser 1989, p. 84). The mathematization of a word problem is seen in this respect as a stepwise, word-based and knowledge-based process of construction and modelling "from the text to the situation to the equation"—and back. In accordance with the model, first of all, based on the understanding of the wording of the problem and with the help of individual knowledge of the world, an *episodic situation model* is constructed. This situation model is transformed into a *mathematical problem model.* Within this, the conceptions from the situation model are reduced in abstract form to a problem-relevant semantic framework with a mathematical correspondence. Compared to the mere episodic understanding of an (everyday) situation, only those elements which are necessary for the mathematical problem solving are retained. In a subsequent further reductive mathematization step, a solution equation is then worked out. The procedural solving of this through arithmetic operations leads to a *numerical value* which, interpreted according to the situation, flows into an answer sentence.

The model from Blum et al. (2004) has a mathematical didactics perspective, which also focuses on the mathematical modelling process as a whole and understands this as a cycle. The starting point for the modelling is either a real-life or non-real situation, which as a rule, due to the high level of complexity, is simplified, idealised and presented in a structured way. In this way, a "*real model*" (Blum et al. 2004, p. 48) is produced, the mathematization of which leads to the "*mathematical model*" (ibid). The mathematical model is worked on in this respect using heuristic strategies and mathematical knowledge. This working process leads to "mathematical results" (ibid). These have to be interpreted in terms of the real situation and lead to "real results" (ibid), which have to be checked and validated against the real situation.

The two models, which are compatible in terms of their structure, have in common the complexity of their approach, cognitive-constructivist perspective, and the necessity to really *understand* what the situation is about, i.e. to construct an appropriate

[3]The cognitive psychological model was developed in 1985 as a computer simulation model and was programmed in LISP (Reusser 1985, 1989). The executable model, which was programmed on a XEROX Lisp machine from the early 1980s, solved several dozen elementary mathematical word problems. In a language-based coding and mathematization process, which comprised up to 150 construction steps and micro-inferences, the linguistic input (the word problem) was transformed, through an episodic and mathematical situation model, into an equation and subsequently a mathematical answer sentence.

situation model, and mathematize in a stepwise fashion when solving a task. Both models emphasise the understanding of the action and factual situation underlying the task as a necessary intermediary step, or rather as an indispensible act of mental construction in terms of a deeper mathematical understanding of the task. However, what does understanding mean in a concrete individual case, and what are its components?

When understanding and working on a word problem, central elements of problem-setting and the described factual situation must be ascertained and related to each other. These important elements for understanding a situation or a concept are named here, in line with Drollinger-Vetter (2009), as "*elements of comprehension*". The concern is with content-specific elements of meaning, with central insights, which are necessary in order to work on the problem.

With regard to concrete modelling processes, this means that on the level of real-world situation conceptions, identifiable elements of comprehension are linked to a mathematical problem conception and finally an equation. After carrying out operations and procedures, this equation then leads to a numerical answer, and this numerical answer can in turn be interpreted in relation to understanding.

However, as mentioned in the introduction, numerous findings make it clear that learners often tackle word problems too hastily on the numerical linking level. In other words, they try to solve the task arithmetically without having tried to understand the problem situation as part of the real world and without having dealt with the factual content. Verschaffel et al. (2002, p. 265) speak in this regard of "the game of word problems", and refer to the implicit and explicit game rules and norms, according to which learners solve word problems in school—rules which indicate a barely reflected upon "a socio-cognitive grammar of school-based problem-solving" (Reusser 1988, 1999) that often has little orientation towards understanding.

These and similar findings on solving problems *without a deeper understanding*, have frequently led to traditional, "closed", often artificial-looking word problems being fundamentally called into question and being replaced by more complex, realistic modelling tasks. Nevertheless, many traditional word problems require a situation model to be constructed for their understanding-based solution, which can subsequently be transferred to a mathematical model. However, so that innovative and traditional word problems indeed become learning opportunities for authentic problem-solving and modelling, learners must be supported through explicitly focusing on the development of a situation understanding and the mathematization that emanates from this. In other words, understanding has to be oriented towards the specific elements of comprehension which are constitutive for understanding the task, both on the level of the situation understanding and on the level of mathematization. A more in-depth understanding of a mathematical word problem comprises both the understanding of the problem on the level of the action situation or factual situation and on the level of its mathematization as an expression of an extended symbolisation.

2.2 Classroom Discourse as a Framework for Didactic Communication

In the context of reform programmes, in addition to a new task culture, there are calls for a renunciation of the traditional questioning-developing instruction in favour of an arrangement of learning environments aimed more strongly at self-guided learning and problem-solving, such as cooperative solving of problems in small groups (cf. Leiss 2007). However, in everyday instruction, the mutual solving of word problems in classroom dialogue continues to play an important role. This was also shown in the current video study: In 35 of the 37 classes examined, the problem being analysed was solved partially or completely within the framework of a classroom dialogue.

From the theoretical perspective of situated learning, the quality of the discourse, particularly with regard to the building up of a flexible and productively usable knowledge base and of necessary problem-solving competencies and dispositions, seems important (cf. amongst others, Greeno 2006a): The active role of the learners in the dialogue as co-producing, co-responsible participants in co-constructive processes of generating, questioning and evaluating ideas is seen from this viewpoint as a central prerequisite for building up knowledge that can be flexibly implemented and which considers subject-specific problem-solving competences.

Numerous descriptions can be found in the literature of productive classroom discourse, which activates the learners to a higher degree (Lampert and Cobb 2003; Leinhardt and Steele, 2005). What these descriptions have in common is their focus on the positioning or role of the learner in the teacher-student interaction, in terms of a transition from mere questions and answers or keyword-giving, to participants with equal rights, who are jointly responsible in the generation of problem solutions and knowledge. In these cases, learners do not merely *react* with (mostly short) answers to (mostly narrowly posed) questions or impulses of the teacher, which are then evaluated by the teacher. Rather, they bring in substantial ideas or suggestions themselves, substantiate and support their position, react to counter-proposals, appraise the conclusiveness or productivity of these in relation to the problem solution. Greeno (2006b, p. 538) described this changed role as "*authoritative and accountable positioning*". Altogether, it becomes clear that aside from the actual subject content, it is also always a matter of the way in which the discourse about this content is led and of the participation of the learners.

Against this background, the video analyses will also measure the extent to which the students were able to participate in the discourse with substantial contributions and in this way help to steer the problem-solving process. However, as yet there are barely any analysis instruments that capture a systematic, quantitative measurement of subject-didactically relevant, content-related aspects and at the same time take into account to extent of opportunities for the students to participate. The development of a respective instrument therefore constitutes an important partial goal of the current video study.

In the current article, selected results of the analysis undertaken are reported. These relate to the following research questions:

- How do teachers support the process of understanding with regard to content-related, mathematical aspects? In other words, to what extent do they work on central, content-based elements of comprehension explicitly in the sense of supporting the situation understanding and/or the mathematization?
- To what extent does the instrument developed for instructional coding make visible important differences with regard to the theory? Can possible profiles or types be described?

- To what extent can students participate in classroom discourse with substantial contributions?

3 Method

The data set used for the analysis originates from the Swiss-German project "*Quality of instruction, learning behaviour and mathematical understanding*" (Klieme et al. 2006b, 2009), which was conducted by the German Institute for Educational Research (Deutsches Institut für Pädagogische Forschung; DIPF) and the Institute for Educational Science of the University of Zurich. The aim of the project is "to examine components of instructional quality, in particular relations between didactic behaviour, mathematical learning achievement (understanding) and personal variables (interest, learning strategies, attitudes)" (Reusser and Pauli 2000, p. 31).

The project ran from 2000 to 2006 and was structured into three phases, each of which lasted two years. These three phases had different research focuses: In the first phase, the concern was with a representative survey of teachers concerning instruction-related, self-related and school environment-related cognitions. The second project phase addressed the video-based recording of two different instructional modules (Pythagoras module and word problem module) in 20 German classes of the 9th school year and in 19 Swiss classes of the 8th school year. In the third project phase, an Internet-based training programme for the teachers was carried out.

The lessons used for the current investigation come from the second phase of the project.

3.1 Data Set

The sample consisted of 37 teachers and their classes. This study deals with classes from the 8th or 9th school years from the highest track (Gymnasium) and from the middle track ("Realschule", "Sekundarschule") of the three school types in Germany and Switzerland. 18 classes are from Germany, and 19 are from Switzerland.

In order to enable as representative an insight as possible into the everyday working on word problems, the teachers were not specifically prepared for participation in the study. Several days before the arranged double lesson, they were sent word problems which had to occur in the instructional unit. Furthermore (e.g. for the methodological-didactic arrangement of the instruction when working on the problems), with one exception[4], no guidelines were laid down.

For the analysis presented in the following, one of the tasks set was selected. The object of the analysis is the public working on thus problem. This ensued in the form of classroom discourse, but also in the form of publicly given learning support.

[4]Exception: The teachers were asked to carry out at least one instance of group work at some point in the course of the instructional unit.

3.2 Task and Task Analysis

The task is one that is typical of everyday mathematics instruction, of relatively low complexity and similar to those found in mathematics teaching aids:

"Marie is now twice as old as Anna. Peter is half as old as Anna. Four years ago, however, Marie was six times as old as Anna. How old are Anna and Marie now?"

As a prerequisite for the development of the analysis instrument, first of all, a task analysis was carried out. In this respect, among others, the following four central elements of comprehension were identified, which are imperative for understanding the problem in terms of constructing a situation model and transferring it to a mathematical problem model:

1. Relation between Anna's age and Marie's age, which can be formulated at a specific point in time.
2. Irrelevance of the statement regarding Peter's age.
3. Chronological progression of the formulation of the problem: It must be understood that the problem describes a situation at different points in time.
4. The relation of Anna's age to Marie's age changes over time; it is a matter of a discrete number of years which pass and which affect each individual age in absolute terms. However, the relation between the two ages fundamentally changes through this and does not remain stable.

3.3 Video Analysis

The aim of the video analysis was to describe the mutual problem-solving process in classroom discourse in terms of the understanding-oriented support of the problem-solving as well as the extent of opportunities for student participation in the discourse through substantial contributions. In terms of the operationalisation of the individual categories and sub-categories for the coding, we applied our own theory-based developments, which were established among other things on the basis of a detailed subject-didactical task analysis following Neubrand (2002; for more detailed information, cf. Brunner 2008). The videos were coded in parallel by two raters. A consensus was reached in cases of differing evaluations.

3.3.1 Understanding-Oriented Support: Explicit Addressing of Situation Understanding vs. Mathematization

The content-related analyses ensued with an analysis instrument developed for that purpose. Essentially, the analysis was aimed at the question of the extent to which, when solving the problem, on the one hand the construction of a situation understanding (situation model) and on the other hand the mathematization thereof were explicitly addressed. In order to establish this, the dialogues were initially searched for instances of addressing elements of comprehension (cf. Sect. 3.2). For each of the identified elements of comprehension, it was established on which level of understanding it was addressed: on the level of the situation understanding and/or on that of mathematization.

In the example of element of comprehension 1, the respective criterion which was checked on the level of situation understanding was as follows: "The real-life relation between Anna's age and Marie's age, which can be formulated for a particular point in time, is addressed". On the level of mathematization, by contrast, the criterion was: "The relation between Anna's age and Marie's age is implemented as an equation".

For each of the four elements of comprehension, analogously to this, criteria were developed for the explicit addressing on the level of the situation understanding and the mathematization in the instruction. Moreover, for individual elements of comprehension, a total of five sub-categories were differentiated, which were also measured in the implementation as situation understanding and as mathematization. One of these sub-categories, for instance, was the measurement of the correct variable description. Thus, it was established whether a variable was equated with the object or with a characteristic: In other words, was the talk of "x is Anna" or "x is *the age* of Anna"?

Element of comprehension 2, the irrelevance of one of the pieces of information regarding age, assumes a particular importance here: On the level of the situation understanding, it was examined whether or not it is explicitly addressed that this piece of information was irrelevant for the problem-solving process. The mathematization of this element of comprehension, by contrast, meant that in the class, Peter's age was also implemented as an equation.

3.3.2 Student Participation in Classroom Discourse: Communication Patterns in Addressing the Elements of Comprehension

A second focus of the analysis, and consequently of the development of the respective analysis instrument, was the question of to what extent the learners actively participated in addressing the elements of comprehension through substantial contributions. To this aim, each time an element of comprehension was addressed in the instruction, it was rated in terms of its extent of student participation. The ratings were allocated to one of four communication patterns.

Communication pattern 1 corresponds to an exemplary demonstration or an explanation without any active participation of the learners. Communication pattern 2 corresponds to a narrowly driven teacher discourse which consists of repeated "initiation-reply-evaluation" (IRE) sequences (Mehan 1979), with the students merely able to participate in the role of keyword givers. An example for the implementation of mathematization of element of comprehension 1 in communication pattern 2:

T: Now, what do you suggest? Michael.[5]
A: Erm, Anna is X.
T: Anna, now: X, hm.
A: Yes.
B: Erm, Marie: Two X.
T: Erm [yes]. . . . Marie is twice as old as Anna. . . . Peter is half as old as Anna.

[5]T: Teacher; A–C: Students.

C: One half.
T: Yes, or X half.

Communication pattern 3 contains a substantial student participation in a teacher-led discourse. Communication pattern 4 is characterised by a co-constructive participation of the students in terms of a community of learners, which corresponds more to a discussion among equals than to a teacher-driven discourse, such as in the following example:

T: No questions?
A: Tobi.
B: Well, you should have—erm—on the left side been able to calculate X minus four, put it in brackets and multiply it by six. Why did you just calculate the right side by one sixth?
A: Well… why… Well, do you have another suggestion for me? Would you have another suggestion?
B: No, well, just, why did you—well, times one sixth, that's ok but why did you just take one sixth and not put the left side—well in brackets and multiply it by six. That would be would have been simpler.
A: Ah, the other one. You mean there, if I put it in brackets times one sixth?
B: Mm [Yes].

3.3.3 An Overview of Further Analyses

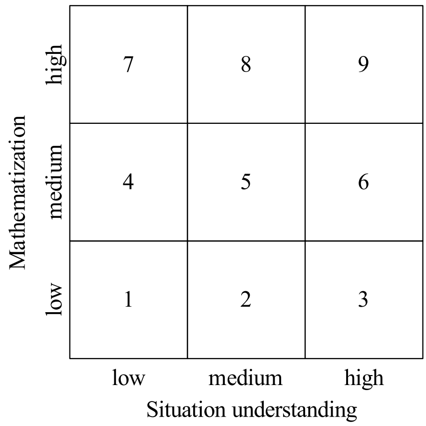
In addition to the measurement of elements of comprehension described above (situation understanding vs. mathematization; communication pattern), further features of supporting the problem-solving process were measured which are relevant for the quality of understanding-oriented support of the problem-solving process from a subject-didactical and/or cognitive psychological perspective. These include, among other things, the *procedural implementation* of the elements of comprehension in the rearranging of terms and the solving of equations, the *application of heuristic strategies and heuristic aids* as well as *didactical support possibilities for the problem-solving process*. These entail, besides other aspects, the precise terminologies, the type of language, the embedding into subject matter, making explicit the question that has been posed, measuring the variable term used, addressing the type of task, and the question of the openness of the problem and solution sphere. Moreover, some *information on the context of the lesson* is important, for instance the type of arrangement of the instruction, the duration of the work and possible subject-based errors.

These analyses were evaluated in descriptive terms. As the corresponding evaluations cannot be presented within the framework of this article, we will not address them any further here (for more detailed information, see Brunner 2008). Here, we limit ourselves to the presentation of the results on two themes: (1) *elements of comprehension*, and (2) *student participation in the problem-solving process*.

3.4 Profile Building

Based on the theoretical notions regarding the solving of modelling tasks (cf. Sect. 2), it was particularly interesting to assess whether the central elements of comprehension explicitly appeared and were worked on during the classroom discourse. Ideally,

Fig. 1 Nine different profiles of understanding-oriented instruction



one could assume that in an optimally supportive, understanding-oriented solution process, all central elements should be covered, both with respect to the real-world situation understanding and to the mathematical modelling. In this case, the modelling process would be conducted completely, and implemented in a balanced manner regarding the situation understanding and the mathematization. It was, however, presumed that the teachers would perhaps set different focuses and that for this reason, incomplete solution processes would also arise. A profile building was undertaken in order to check this assumption.

The data-driven approach enabled the establishment of three areas of high, medium and low extent through counting the number of elements of comprehension and sub-categories that arose. A complete working of all elements of comprehension including sub-categories corresponded to 9 points. A high extent of situation understanding or mathematization was set at 6–9 points, a low extent was seen as lying in the area of 0–2 points, and a medium extent in the area of 3–5 points.

A matrix with nine fields was chosen in order to describe the profiles. Each field describes a different degree (low, middle, high) of the situation-based (situation understanding) elements of comprehension, and those occurring in the course of the actual mathematization (cf. Fig. 1).

The profiles can be understood as a systematic description of an instructional practice, and do not tell us anything about a possible adaptive behaviour of the teacher in dealing with word problems. Nevertheless, it can be established that from a theoretical perspective, above all profiles 4 and 7 appear rather problematic, because the situation understanding is neglected compared to the mathematization. Similarly problematic is profile 1, in which there is only a low extent of both situation understanding and mathematization.

4 Results

The results are presented in relation to the research questions posed following Sect. 2. The first section is concerned with the results regarding the content-related support

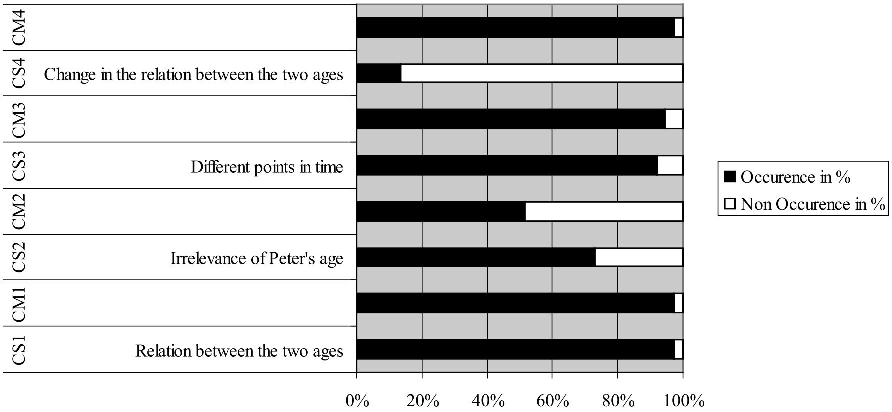


Fig. 2 Percentage share of classes in which an element of comprehension (CS: element of comprehension on situation level; CM: element of comprehension on mathematization level) occurs or does not occur; $N = 37$

by teachers, i.e. the explicit addressing of elements of comprehension. The second section demonstrates the extent to which it was methodologically possible, using the developed instrument, to describe relevant differences between the classes with regard to the modelling process. In the third section, the possibilities of the students to participate in the problem-solving process are presented.

4.1 Explicit Addressing of Elements of Comprehension

In relation to the modelling process, in the current work, four central elements of comprehension of the age task were described, which were formulated as *elements of comprehension of the situation and of mathematization*. On the whole, it was apparent that the elements of comprehension that are described as central were not explicitly tackled in all classes (cf. Fig. 2).

Element of comprehension 1—the relation of the two given ages to one another—was addressed on the situation level and mathematized in 97.3% of all cases. However, the concern here was not with the same cases, i.e. in one class the element of comprehension was discussed on the level of the situation understanding, but was not mathematized. In another class, it was mathematized, but was not clarified on the level of the situation understanding.

Element of comprehension 2—addressing the irrelevance of an additional piece of age information—was discussed in 73% of all cases. In the remaining 27% of classes, element of comprehension 2 was not addressed on the level of situation understanding, i.e. in these cases, the irrelevant piece of age information was implicitly accepted, but it was not explicitly illustrated *that* the information was irrelevant. In 51.4% of cases, element of comprehension 2 was also addressed during the mathematization process. Although the concern here was with the solution of an irrelevant piece of information, in a good half of classes it was therefore also expressed as an equation or term.

Element of comprehension 3—addressing different time points in the task-setting—occurred in 91.9% of all cases on the level of the situation understanding and was also mathematized in 94.6% of all cases.

A quite different picture was apparent in terms of working on element of comprehension 4—addressing the change in the point of time: This was clarified on the level of the situation understanding in only 13.5% of cases, but was mathematized in 97.3% of classes.

Moreover, it is striking in terms of a sub-category of element of comprehension 1 that the correct variable description was only undertaken in the minority of lessons: Only 29.7% of teachers described, for example, the variable x consistently as “age of Anna”, while the vast majority spoke of “$x =$ Anna” or “x is Anna”.

4.2 Description of Relevant Differences Between the Classes: Different Profiles

On the basis of the elements of comprehension for the situation and the mathematization thereof, a profile building was then undertaken (cf. Sect. 3.4) which describes the extent of the explicit addressing of the two aspects. 36 of the 37 classes examined could be allocated to one of the profiles. On the whole, six of the nine possible profiles arose. One class had to be excluded from the allocation to a profile because the teacher transferred a part of the work on elements of comprehension into the homework phase, and these could therefore not be observed.

Twenty-four teachers, i.e. two thirds of all of the cases, could be allocated to a moderate medium profile, in which both the situation understanding and mathematization occur to a medium extent. Eight teachers showed a slight over-emphasis of one category compared to the others: Six of these teachers—and thus a sixth of all cases—emphasised the mathematization strongly, but the situation understanding, on the other hand, to a medium extent, whilst two teachers showed the reverse pattern. One teacher belonged to an extreme group, in which both categories were equally weighted, but only to a low extent (cf. Fig. 3).

Two teachers were found who could be allocated to an actual *top group*: These teachers showed the most or even all of the items for situation understanding and mathematization, and thus a virtual complete explicitly conducted modelling process. It is particularly interesting that for these two teachers, a positive characterisation was also observed relating to a whole array of further features, such as in the use of correct mathematical language and terminology.

Of the twelve teachers who did not belong to the moderate medium profile, ten showed a high extent in at least one of the two areas, of situation understanding and mathematization. Only two teachers belonged to profiles with a low extent of situation understanding and/or of mathematization. Thus, from a theoretical perspective, only a very small group showed a problematic profile. The vast majority of teachers conducted the modelling process to a medium or in part even high extent. The majority of teachers (75%), moreover, could be allocated to a profile with a balanced modelling process—albeit to differing extents.

4.3 Participation of the Students in the Problem-Solving Process

If one considers the didactic communication in general, a predominance of communication pattern 2 was apparent, i.e. a pattern strongly dominated by the teacher in

Fig. 3 Degree of the 9 different profiles of understanding-oriented instruction with the empirically found number of cases ($N = 36$)

Mathematization \ Situation understanding	low	medium	high
high	0 cases	6 cases	2 cases
medium	1 case	24 cases	2 cases
low	1 case	0 cases	0 cases

terms of "Initiation-Reply-Evaluation sequences". This was decisive in 56% of all work steps measured. Communication pattern 1, i.e. a teacher discourse or explanation, was also strongly represented, with 35%. The other two communication patterns, which are characterised by a higher level of learner participation, were rather rare. Indeed, communication pattern 4 was only observed in 1% of all cases and in only two teachers, while communication pattern three still occurred in 8% of all work steps.

Between the elements of comprehension, no differences were apparent in terms of the communication patterns; in the vast majority of cases, all were discussed in communication pattern 2, i.e. the pattern dominated by the teacher in the sense of "Initiation-Reply-Evaluation" sequences. By contrast, in the two teachers who could be allocated to a top profile, there was also an increased extent of communication patterns which demand more participation on the part of the learners.

5 Discussion

Based on the research questions, in the following chapter, three thematic areas will be highlighted: (1) the content-based support of the students through the explicit addressing of the elements of comprehension, (2) methodological aspects for the description of relevant differences between the classes, and (3) opportunities for the students to participate in the problem-solving process.

5.1 Content-Specific Support: Elements of Comprehension

Regarding the processing of the central relations and facts identified for the task—which are called here "*elements of comprehension*" as in Drollinger-Vetter (2009)—the findings make it clear that the classes differ greatly in this regard. For instance, there are classes in which only some of the elements of comprehension occur or in which the elements of comprehension are a topic on the situation level but are not mathematized; or are mathematized, but not addressed on the situation level. Against the background of theoretical foundations and empirical findings on solving word problems presented in Sect. 2, particularly the latter would seem rather problematic. If the establishment of a situation understanding is foregone in favour of a direct mathematization, it should at least be asked whether the instruction contributes to consolidating and using suboptimal strategies by the learners, for instance through the fact that a deeper understanding of the task itself is replaced by an orientation towards keywords (cf. Schoenfeld 1982) and thus towards features of the linguistic surface.

The fact that in the majority of classes, not all elements of comprehension were addressed explicitly and thus the modelling process was only explicitly accomplished in part, however, also gives rise to the question of to what extent a *complete* explicit processing of all elements of comprehension is even necessary. To clarify this, in the framework of an additional analysis of the data, it would also have to be examined whether, and to what extent if need be, elements of comprehension were worked on in the students' individual work and consequently whether a virtually complete processing of the individual elements of comprehension has taken place. Moreover, in the sense of an adaptive instruction, it is possible that in certain classes it appears functional to place a main focus on the situation understanding or the mathematization. In the allocation of the profiles, it was not possible to incorporate differences in the classes regarding level of demand or average ability. Consequently, even a moderate profile could depict a meaningful didactic working on a word problem in an adaptive sense. However, in any case, those profiles would remain problematic which show a low extent in the situation understanding or mathematization, as this might not only be the expression of an imbalance between situation understanding and mathematization, but also in particular might represent the neglect of at least one important part of the modelling process. If the situation understanding is neglected, it might be feared that behavioural patterns such as "superficial modelling" (cf. Sect. 1.2) become stabilised, and unfavourable problem-solving strategies—irrespective of the quality of the task—might persist.

The fact that there are teachers who, from a theoretical perspective, can be allocated to a top profile, which is characterised by the completeness and balance of the modelling process, and that these teachers above and beyond this also differ from the middle group in other characteristics, supports the assumption of the importance of the (near) completeness of the modelling process for qualitatively high-level, understanding-oriented mathematics instruction. In these classes, "adaptive expertise" (cf. Hatano 2003) could be fostered. However, to clarify this question, it would also be necessary to include test data which measure the learning success of the students. For the analysis presented here, such data are not available.

5.2 Analysis Instrument for the Description of Differences Between Classes

By breaking down the explicit addressing of elements of comprehension on the level of situation understandings and mathematization, the analysis instrument developed for the current study contributes towards precisely describing classes in terms of how they work on the modelling process. In this way, different profiles can be captured, which differ from one another in terms of the extent of the explicit tackling of a situation understanding and/or mathematization. As the modelling of problem-solving tasks is seen as demanding, it is important to establish the extent to which mathematization occurs in a class, or to which a solid situation understanding is developed. Moreover, the profiles demonstrate a relevant opportunity for optimization for the modelling process. In the framework of reform efforts concerned with improving the quality of tasks (cf. Sect. 1.1), the instrument might be implemented both for the evaluation of the tasks and their optimization as well as to reflect upon instructional implementations. The formulated, subject-didactically relevant criteria for the process of working on a task, as they are undertaken in the analysis instrument, can be understood as a content-based level of expectation or possibility of didactic support. The coupling of content-based solution step and communication pattern enables a further dimension of the optimisation processes. Thus, the instrument can be used both for the measurement and the reflection of instructional processes.

The profiles are conceived in a two-dimensional manner, in that they are based only on the situation understanding and the mathematization. Further, equally relevant support possibilities are captured with the instrument, but they are not incorporated in the profile-building. The question also remains open of whether or not a particular profile is functional in an adaptive sense of teacher behaviour. To ascertain this, it would be necessary to include data on the level of demand and the student ability.

5.3 Participation of the Students in the Problem-Solving Process

The modelling process—be it nearly complete or conducted partially—takes place in the discourse, in the interaction between learner and teacher, within the framework of the classroom discourse which has been studied here. Although the preference found here for the communication pattern 2, and thus an instructional discourse dominated strongly by the teachers in terms of "initiation-reply-evaluation" sequences, comes as no real surprise due to the findings of other studies, several questions from the perspective of situated learning nevertheless arise. If the quality of the discourse in terms of building up a flexibly usable knowledge base and central problem-solving competences is to be considered important (cf. Greeno 2006a), then there is potential for development in numerous classes, less in relation to the execution of the modelling process than in relation to didactic communication.

However, the importance of student participation is not only central to the classroom discourse which is led by the teacher. But it is also equally central to reform-oriented (more open) ways of arranging instruction, as well as for advisory or supportive dialogues during independent or group work phases. Moreover, it is also important in relation to a learning culture which stimulates the students to critically question tasks and carry out problem-solving rather than simply working through tasks that have little meaning. This is why didactic communication, relating to actual subject content, is also due appropriate attention in teacher education and (further) training.

5.4 Conclusion

In the work presented here, it was shown how a modelling process (which is theoretically postulated, seen as important for understanding and solving mathematical word problems, and multi-staged in nature) can be measured in terms of its subject-content-based and communicative qualities as a didactic support and communication process.

Although it was not possible to draw reference to performance data in the current data set (cf., however, Drollinger-Vetter 2009), the findings demonstrate the mathematical-didactical potential and the sensitivity of an analysis in which mathematics-related processes of understanding can be reliably measured beyond general-didactical categories. Such a measurement, which, moreover, also includes quality features of didactic communication, is of importance and practical use both from a (methodological) research perspective and from perspectives of teacher training.

In terms of all of the findings presented here, it must be taken into account that on the whole, the sample is rather small. As a further limitation, it must be noted that it was solely a matter of working in the framework of whole-class instructional discourse. It is possible that precisely this form of classroom discourse entails numerous didactic difficulties, for example on the level of learner participation, so that the question needs to be asked whether the classroom discourse type of instructional arrangement does not differ fundamentally from other instructional arrangements and whether a communication pattern is thus more dependent on the instructional arrangement in the lesson or on the actual task. Analyses relating to this are underway. The two teachers who were allocated to a top profile show, however, that a challenging working is also possible in the framework of classroom discourse.

From the methodological point of view, an analysis instrument was successfully developed which covers both mathematical content elements as well as the participation structure of communication in the instruction. To what extent the instrument can be transferred to other problem types is currently being examined in the framework of other analyses. Should it become apparent that the analysis instrument developed here can also be implemented for other types of problem, then the question could be clarified of whether the support behaviour of teachers in the sense of the profiles presented here can be described more as rather stable, personal profiles, or whether the categorisation into profiles depends on the task being worked on.

In terms of subject didactics, didactic communication must be aligned towards understanding and the elements thereof. In terms of dialogue and communication, it should be oriented towards co-construction, student participation and autonomy support. Accordingly, the quality of support should be understood in terms of subject didactics and from the perspective of dialogue and communication. Therefore, it should also be analysed as such.

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